The complexity of energy eigenstates as a mechanism for equilibration

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Understanding the mechanisms responsible for the equilibration of isolated quantum many-body systems is a longstanding open problem. Some of them have been identified, but a complete picture is still missing. In this work we obtain a link between equilibration and the complexity of the hamiltonian's eigenvectors. We see that hamiltonains with generic eigenvectors equilibrate, and provide a simple closed formula for the equilibration time-scale. This allows us to obtain the equilibration time-scale for random hamiltonians. We also attempt to quantify the complexity of the energy eigenvectors, by considering hamiltonians whose diagonalizing unitary is a quantum circuit. We show that when the circuit size is quadratic or larger equilibration is expected, and when it is linear or smaller equilibration is not expected.

INTRODUCTION

A physical system that has been sitting for a while is often described by a thermal or Gibbs state. This presupposes that, whatever the initial state was, the system evolved into a stationary state. But this is impossible for closed systems evolving unitarily, unless the initial state was already stationary. However, often, the reduced density matrix of a subsystem does evolve to a stationary state, and stays close to it for most of the time —this is called local equilibration. Identifying which conditions are responsible for this process is a long-standing question in Physics, both for classical [1, 2] and quantum systems [3–5]. Recently, significant advances in the understanding of this problem have been achieved due to three factors. First, powerful numerical techniques have enabled the dynamical simulation of large many-body systems [6, 7]. Second, the use of quantum-information ideas, and in particular of entanglement theory, has provided new perspectives into this question [8–10]. Finally, experiments with ultra-cold atoms have allowed the manipulation and observation of many-body systems, with high control [11]. Interestingly, these experiments have challenged the current understanding of these questions, since no thermalization was observed in non-integrable systems [12, 13].

Local equilibration can be explained by dephasing [4, 8, 14, 15], under the condition of no degenerate energy gaps (no resonant transitions) [16]. This mechanism could be the complete answer, but it turns out that it does not explain all equilibration processes. Systems of quasi-free bosons have infinitely-many degenerate energy gaps, and some of them enjoy local equilibration [9, 17, 18].

In this work, we consider a different mechanism for local equilibration, which is complementary to dephasing since it is based on the structure of the energy eigenvectors instead of the eigenvalues. This mechanism is independent of the energy spectrum, hence it allows to construct hamiltonians which enjoy local equilibration despite having many degenerate energy gaps. Additionally, this mechanism gives reasonable equilibration time scales, while the condition of no-degenerate energy gaps alone gives unrealistically large values [8, 19].

Our main goal is to explore the relationship between the equilibration properties of a system and the complexity of its energy eigenvectors. First, we show that hamiltonians with generic eigenvectors enjoy local equilibration, independently of their spectra and whether they have degenerate energy gaps. Second, we show that the dynamics of the convergence to equilibrium and its time scale do depend on the spectrum, in particular, on the Fourier transform of the level density. We calculate this function for a random matrix and for the spectrum of an integrable system, obtaining their corresponding equilibration time-scales. Third, we attempt to quantify the complexity of the eigenvectors of a hamiltonian and relate this to its equilibration properties. In order to do so, we consider all hamiltonians whose diagonalizing unitary is a quantum circuit with a fixed number of gates. We show that, for almost all circuits with number of gates quadratic in the system's size the associated hamiltonain enjoys local equilibration. Also, for almost all circuits with less than linear number of gates the associated hamiltonian does not equilibrate.

EQUILIBRATION IN HAMILTONIANS WITH GENERIC EIGENSTATES

We consider a system of N qubits (spin-$\frac{1}{2}$ systems) with hamiltonian

$$H = \sum_{n=1}^{d} E_n \left|\Psi_n\right\rangle\!\left\langle\Psi_n\right| = U \begin{bmatrix} E_1 & & \\ & \ddots & \\ & & E_d \end{bmatrix} U^\dagger, \qquad (1)$$

where $d = 2^N$. The hamiltonian characterizes the dynamics of the system: if ρ is the state of the N qubits at time $t = 0$ then $\rho(t) = \mathrm{e}^{-\mathrm{i}tH}\rho\,\mathrm{e}^{\mathrm{i}tH}$ is the state at time t. The diagonalizing unitary U maps a local basis

$|n\rangle = |n_1, \ldots, n_N\rangle$ to the energy eigenbasis $|\Psi_n\rangle = U|n\rangle$, for all $n \in \{0,1\}^N$. We do not impose any constraint on the spectrum of the hamiltonian, and in particular, it can have many degenerate energy gaps [16]. The maximal degeneracy of the hamiltonian is denoted by $g = \max_n |\{n' : E_{n'} = E_n\}|$. We will show that, in hamiltonians with generic eigenstates, the dynamics of the convergence towards equilibrium is characterized by the Fourier transform of the level density $\mu(E)$,

$$\tilde{\mu}(t) \;=\; \int dE\, \mu(E)\, \mathrm{e}^{\mathrm{i}tE} \;=\; \sum_{n=1}^{d} \frac{1}{d}\, \mathrm{e}^{\mathrm{i}tE_n}\,, \qquad (2)$$

where the second equality holds for finite-dimensional systems.

Suppose that we are interested in a subset of the N qubits—we refer to it as the *subsystem*, while the rest of qubits are referred to as the *environment*. The Hilbert space factorizes as $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, with the corresponding dimensions satisfying $d = d_S d_E$. If ρ is a state of the N qubits, the reduced state of the subsystem is $\rho_S = \mathrm{tr}_E \rho$. *Local equilibration* happens when the subsystem evolves towards a particular state, and stays close to it for most of the time. If the state $\rho_S(t)$ equilibrates to the state $\bar{\rho}_S$ then, this must be the time-averaged one [10]

$$\bar{\rho}_S = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \rho_S(t)\;. \qquad (3)$$

As in [8–10], we quantify the departure from equilibrium at time t by the trace distance $\|\rho_S(t) - \bar{\rho}_S\|_1$, which is directly related to the probability of distinguishing $\rho_S(t)$ from $\bar{\rho}_S$ with the optimal measurement [20, 21]. The following result (proven in Appendix A) bounds the departure from equilibrium in hamiltonians with generic eigenstates.

Result 1. For any N-qubit initial state ρ, almost all hamiltonians with a given spectrum $\{E_1, \ldots, E_d\}$ satisfy

$$\|\rho_S(t) - \bar{\rho}_S\|_1 \leq\; \frac{d_S^{1/2}}{\epsilon} \left(|\tilde{\mu}(t)|^4 + \frac{g^2}{d^2} + \frac{6}{d_E} \right)^{1/2} \qquad (4)$$

for all t.

The meaning of "almost all" is controlled by the free parameter $\epsilon \in (0,1)$, which is an upper bound for the fraction of hamiltonians that violate the bound according to the Haar measure [22]. For example, if we set $\epsilon = 0.01$ then 99% of the hamiltonians satisfy the above bound [23]. Note that this bound is independent of the initial state ρ, but the set of hamiltonians which violate it could depend on ρ.

Let us discuss the significance of the three terms inside the brackets of (4). The first term depends on the spectrum of H and the time t. At time $t = 0$ the bound is useless, since $\tilde{\mu}(0) = 1$. For sufficiently long times, it is expected that the phases in the sum (2) cancel each other, resulting in a small number, and implying equilibration. Actually, Result 2 below shows that this is the case for most of the times, independently of the existence of degenerate energy gaps. However, in finite systems ($d < \infty$), there are some very special (and very long) times t_{req} for which $\tilde{\mu}(t_{\mathrm{req}}) \approx \tilde{\mu}(0)$. These are the quasi-recurrences, in which the system goes back to a non-equilibrium state. In the thermodynamic limit quasi-recurrences tend to disappear. In Section , the quantity $\tilde{\mu}(t)$ is calculated for some meaningful spectra. The second term also depends on the spectrum of H, and implies that hamiltonians with high degeneracy cannot be warranted to equilibrate. The third term implies that equilibration needs the environment to be much larger than the subsystem. This condition is necessary in all approaches to equilibration known to the authors [4, 8–10, 14, 15, 17, 18].

Although quasi-recurrences take the subsystem out of equilibrium, the following result (proven in Appendix A) shows that, in some circumstances, the subsystem is close to the stationary state $\bar{\rho}_S$ most of the time.

Result 2. For any N-qubit initial state ρ, almost all hamiltonians with a given spectrum $\{E_1, \ldots, E_d\}$ satisfy

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \|\rho_S(t) - \bar{\rho}_S\|_1 \;\leq\; \frac{1}{\epsilon} \left(\frac{g}{d_E} + \frac{6\, d_S}{d_E} \right)^{1/2}. \qquad (5)$$

This shows that, in the reasonable regime where $g, d_S \ll d_E$ equilibration is expected for all hamiltonians except for a fraction $\epsilon \in (0,1)$ according to the Haar measure. This establishes the existence of a mechanism for equilibration which is only based on the genericness of the diagonalizing unitary U, or equivalently, the genericness of the energy eigenstates. This mechanism does not rely on any condition for the energy spectrum—even in the presence of many degenerate energy gaps equilibration happens.

TIME SCALES

Result 1 shows that the dynamics of the convergence to equilibrium is given by the function $|\tilde{\mu}(t)|$, which only depends on the spectrum. In this section we calculate $|\tilde{\mu}(t)|$ for two spectra, each representing an extreme case.

Spectrum of a random hamiltonian

Let us consider random hamiltonians sampled from the gaussian unitary ensemble [24]. According to this, each matrix element $H_{ij} = H_{ji}^* \in \mathbb{C}$ is an independent random variable with probability density

$$P(H_{ij}) = \begin{cases} (1/\pi)^{1/2}\, \mathrm{e}^{-H_{ij}^2} & \text{if } i = j \\ (2/\pi)^{1/2}\, \mathrm{e}^{-|H_{ij}|^2} & \text{if } i \neq j \end{cases}.$$

It is shown in [24] that, according to this measure, the eigenvalues of H are statistically independent from the eigenvectors of H, and the diagonalizing-unitary U follows the Haar measure [22], as in results 1 and 2.

The convergence function corresponding to the spectrum of a random matrix is, in the large-d limit,

$$|\tilde{\mu}(t)| \approx \frac{2\, J_1(t\sqrt{2d})}{\pi\, t\sqrt{2d}} \;, \tag{6}$$

where J_1 is the Bessel function of first kind (see Appendix B). This gives an equilibration time-scale

$$t_{\rm eq} = \frac{1}{\sqrt{2^N}} \propto \frac{1}{E_{\rm max}} \;, \tag{7}$$

where $E_{\rm max}$ is the largest eigenvalue of the hamiltonian. Note that both, in terms of N and in terms of $E_{\rm max}$, the equilibration time (7) is much smaller than the one obtained for the spectrum of an integrable system, see Eq. (11) below. This is expected, since the spectrum associated to (7) is maximally chaotic.

Spectrum of an integrable system

Let us consider hamiltonians with the diagonalizing unitary U being generic, and the spectrum being the one of the Ising model in a transverse magnetic field h. (Note that the hamiltonain of the Ising model need not be in this category.) The eigenenergies are parametrized by the vectors $n = (n_1, \ldots, n_N)$, where $n_k \in \{0, 1\}$ are the occupation numbers of the energy eigenmodes:

$$E_n \;=\; \sum_{k=1}^{N} n_k\, \omega(2\pi k/N), \tag{8}$$

$$\omega(\phi) \;=\; \sqrt{(h - \cos\phi)^2 + \sin^2\phi}\;. \tag{9}$$

In the $t \ll 1$ regime we obtain (see Appendix B)

$$|\tilde{\mu}(t)| \approx {\rm e}^{-t^2 N(1+h^2)/8}. \tag{10}$$

This gives an equilibration time-scale

$$t_{\rm eq} = \frac{1}{\sqrt{N(1+h^2)}} \propto \frac{1}{\sqrt{E_{\rm max}}} \;, \tag{11}$$

where $E_{\rm max}$ is the largest eigenvalue of the hamiltonian (see Appendix B).

EQUILIBRATION AND COMPLEXITY

In the theory of quantum computation every algorithm can be represented by a quantum circuit, a sequence of one and two-qubit gates, which processes the input data to generate the output [20]. This is analogous to classical computation, where algorithms can be represented by circuits of logical gates. These gates can be seen as elementary computational steps, so that harder computations require more gates. Following this idea, one can quantify the complexity of a unitary matrix by the size of the smallest circuit which approximates it to a sufficiently good accuracy. It seems natural to quantify the complexity of the eigenvectors of a hamiltonian by the circuit complexity of the diagonalizing unitary U.

Large complexity

A central part in the proof of Result 1 consists of performing an average over all possible unitaries U. However, the averaged expression only contains a fourth power of $U \otimes \bar{U}$. Then, if instead of all unitaries one averages over a subset being a 4-design, the same result is obtained. In [25, 26] strong evidence is provided to the fact that random circuits constitute good approximations to 4-designs. Assuming this result to be true, we can prove prove the following.

Result 3. Suppose that a given initial state ρ evolves under a hamiltonian $H = U{\rm diag}(E_1, \ldots, E_d)U^\dagger$, where U is any circuit with C gates. For almost all such circuits we have

$$\begin{aligned} &\|\rho_S(t) - \bar{\rho}_S\|_1 \\ &\qquad \leq\; \frac{d_S^{1/2}}{\epsilon}\left(|\tilde{\mu}(t)|^4 + \frac{g^2}{d^2} + \frac{6}{d_E} + d^3\, 2^{-\frac{\alpha C}{N}}\right)^{1/2} \end{aligned} \tag{12}$$

for all t.

The meaning of "almost all" is controlled by the free parameter $\epsilon \in (0, 1)$, which is an upper bound for the fraction of circuits with C gates that violate the bound [27]. The constant α depends on the universal gate set, and it is calculated in [25]. Compared to Result 1 there is an extra term inside the brackets, which disappears in the limit $C \to \infty$. When C is quadratic in the number of qubits (or larger),

$$C \geq \alpha' N^2 \tag{13}$$

for $\alpha' > 3/\alpha$, the extra term is exponentially small in N. It is also easy to check that the time average for $T \to \infty$ gives a bound similar to Result 2. Therefore, there seems to be a relationship between the equilibration properties of the hamiltonian and the complexity of its eigenvectors. Note that quadratic scaling is the minimum needed to warrant that in the process of diagonalizing H, all qubits "talk" to each other.

Small complexity

Consider a hamiltonian with no interaction between subsystem and environment: $H = H_S \otimes I_E + I_S \otimes H_E$ where $I_S(I_E)$ is the identity matrix for the subsystem(environment). In this case, the reduced density matrix $\rho_S(t) = \mathrm{e}^{-\mathrm{i}tH_S}\rho_S\,\mathrm{e}^{\mathrm{i}tH_S}$ does not converge to anything, unless it is in a stationary state from the beginning $[\rho_S, H_S] = 0$. Therefore, interaction is necessary for equilibration. The condition of no degenerate energy gaps implies that there is interaction across all possible bipartitions subsystem-environment [8]. It also implies local equilibration, independently of the complexity of the hamiltonian. Therefore, in order to investigate the lack of equilibration, we have to restrict to Hamiltonians with many degenerate energy gaps.

For simplicity, let us consider hamiltonians whose spectrum is that of quasi-free fermions

$$E_n = \sum_{k=1}^{N} n_k\,\omega_k, \tag{14}$$

where $n = (n_1, \ldots, n_N)$, $n_k \in \{0, 1\}$ are the occupation numbers of the energy eigenmodes, and ω_k are the corresponding excitation energies. Let the subsystem be an M-qubit subset of the N qubits, and the environment the remaining $N - M$ qubits. In the case $C = 0$, the eigenbasis is local $|\Psi_n\rangle = |n_1, \ldots, n_N\rangle$, each qubit evolves independently, the subsystem does not interact with the environment, and hence, there is no equilibration. Let us see that this is still the case when the complexity is sufficiently small.

Let us compute the probability that a random circuit for N qubits has no gates involving any of M fixed qubits. The random circuit is generated by repeating the following process C times: uniformly pick a gate from the universal gate set; if this is a single-qubit gate apply it to a qubit chosen uniformly from the N qubits; if this is a two-qubit gate apply it to a pair of qubits chosen uniformly. The probability that no gate is applied to any of the M qubits satisfies

$$p \;\geq\; \left(\frac{N-M}{N}\right)^{2C}. \tag{15}$$

In this event, there is no interaction, and hence, no equilibration. Suppose the complexity is sublinear: $C \leq N^{\nu}$ with $0 < \nu < 1$. If we fix the size of the subsystem M, in the large-N limit we have $p \approx 1 - 2M/N^{1-\nu}$, and then

Result 4. For almost all circuits with sublinear complexity, the associated hamiltonian with spectrum (14) does not enjoy local equilibration.

CONCLUSIONS

In this work we have addressed the problem of equilibration in isolated quantum many-body systems evolving under unitary dynamics. We have pointed out the existence of a mechanism for local equilibration which is based on the complexity of the energy eigenvectors. Under the action of this mechanism the equilibration time scale is given by a nice closed expression, which we could explicitly compute for random hamiltonians.

The relation between equilibration and the complexity of solving the dynamics of a physical system emerging from our results resembles the situation in classical mechanics, where the notion of integrability plays an important role [2]. In fact, there exists a link between equilibration (formalized by weak mixing [2]) and the difficulty of solving the dynamics of a system: classical integrable systems violate weak mixing, while sufficiently chaotic systems satisfy it. Now, if the circuit size is interpreted as the complexity of solving the dynamics of a quantum system, the resulting picture resembles what happens in classical mechanics. Hence, our results may also contribute to the problem of finding a definition for quantum integrability—a proposal in terms of computational complexity can be found in [28].

Dephasing under the condition of no degenerate energy gaps [8, 15] and the complexity of the energy eigenvectors are two independent mechanisms that explain the phenomenon of local equilibration. Are there other mechanisms, apart from these two? Does any of them play a dominant role in natural phenomena?

Note added after completion of this work: results related to the ones presented here have been obtained independently in References [29] and [30].

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Equilibration bounds

Proof of Result 1

Define the linear map

$$\Omega_t[\rho] = \sum_{E_n \neq E_{n'}} \mathrm{e}^{\imath t(E_n - E_{n'})} \mathrm{tr}_E(\Psi_n \rho \Psi_{n'}), \qquad (16)$$

and note that $\Omega_t[\rho] = \rho_S(t) - \bar{\rho}_S$. The sum $\sum_{E_n \neq E_{n'}}$ runs over all pairs of eigenstates $n, n' \in \{1, \ldots, d\}$ with different energies $E_n \neq E_{n'}$. Any $d \times d$ matrix B satisfies $\|B\|_1^2 \leq d\,\mathrm{tr}(B^\dagger B)$, hence we have the bound

$$\|\rho_S(t) - \bar{\rho}_S\|_1^2 \leq d_S \mathrm{tr}_S(\Omega_t[\rho]^2). \qquad (17)$$

Let $\{|s\rangle; s = 1, \ldots, d_S\}$ be an orthonormal basis of $\mathcal{H}_S$, and $\{|e\rangle; e = 1, \ldots, d_E\}$ be an orthonormal basis of $\mathcal{H}_E$.

Next we consider the case where the initial state is pure $\psi = |\psi\rangle\langle\psi|$. Some calculation shows

$$\begin{aligned} & \mathrm{tr}_S(\Omega_t[\psi]^2) \qquad (18) \\ = & \sum_{E_n \neq E_{n'}} \sum_{E_k \neq E_{k'}} \mathrm{e}^{\imath t(E_n - E_{n'} + E_k - E_{k'})} \\ \cdot & \sum_{s,e,s',e'} \langle se|\Psi_n \psi \Psi_{n'}|s'e\rangle\langle s'e'|\Psi_k \psi \Psi_{k'}|se'\rangle \\ = & \langle \mathrm{left}|U \otimes U \otimes U \otimes U \otimes \bar{U} \otimes \bar{U} \otimes \bar{U} \otimes \bar{U}|\mathrm{right}(t)\rangle \end{aligned}$$

where $\bar{U}$ is the complex-conjugate of U, and

$$\begin{aligned} \langle \mathrm{left}| &= \sum_{s,e,s',e'} \langle se, \psi, s'e', \psi, \overline{\psi}, \overline{s'e}, \overline{\psi}, \overline{se'}| \\ |\mathrm{right}(t)\rangle &= \sum_{E_n \neq E_{n'}} \sum_{E_k \neq E_{k'}} \mathrm{e}^{\imath t(E_n - E_{n'} + E_k - E_{k'})} \\ &\cdot \ |n, n', k, k', \overline{n}, \overline{n'}, \overline{k}, \overline{k'}\rangle. \end{aligned}$$

The average of a U-dependent matrix $F(U)$ over the Haar measure [22] is denoted by $\langle F(U)\rangle_U$. It is shown in [25] that

$$\left\langle U \otimes U \otimes U \otimes U \otimes \bar{U} \otimes \bar{U} \otimes \bar{U} \otimes \bar{U} \right\rangle_U = \sum_\pi |\Phi_\pi\rangle\langle\Phi_\pi| \qquad (19)$$

where the index π runs over the 4! permutations of four elements, and the normalized vectors $|\Phi_\pi\rangle = (V_\pi \otimes I)|\phi\rangle_{15}|\phi\rangle_{26}|\phi\rangle_{37}|\phi\rangle_{48}$ are constructed with the maximally-entangled state $|\phi\rangle_{ij} = d^{-1/2}\sum_{n=1}^d |n\rangle_i|n\rangle_j$ between the i^{th} and j^{th} factor spaces, and the unitary V_π which permutes the four left-most factors spaces, and I which is the identity in the four right-most factor spaces. In the following we will calculate $\|\rho_S(t) - \bar{\rho}_S\|_2^2$ exactly, but we will bound each of the terms in the expression to

illustrate its main features. Some calculus shows

$$\begin{aligned}\langle \text{left}|\Phi_\pi\rangle &= \sum_{s,e,s',e'} d^{-2}\langle se,\psi,s'e',\psi|V_\pi|\psi,s'e,\psi,se'\rangle \\ \langle\Phi_\pi|\text{right}(t)\rangle &= \sum_{E_n\neq E_{n'}}\sum_{E_k\neq E_{k'}} \mathrm{e}^{\mathrm{i}t(E_n-E_{n'}+E_k-E_{k'})} \\ &\cdot\ d^{-2}\langle n,n',k,k'|V_\pi|n,n',k,k'\rangle.\end{aligned}$$

The constraints $E_n \neq E_{n'}$ and $E_k \neq E_{k'}$ imply that almost all permutations π satisfy $\langle n,n',k,k'|V_\pi|n,n',k,k'\rangle = 0$. The only permutations π for which $\langle \text{left}|\Phi_\pi\rangle\langle\Phi_\pi|\text{right}(t)\rangle \neq 0$ are analyzed in what follows. For the identity $\pi = (1234)$ we have

$$\begin{aligned}\langle \text{left}|\Phi_{(1234)}\rangle &= d^{-2}\,\mathrm{tr}_S(\mathrm{tr}_E^2|\psi\rangle\langle\psi|) \leq d^{-2}, \\ \langle\Phi_{(1234)}|\text{right}(t)\rangle &= \sum_{E_n\neq E_{n'}}\sum_{E_k\neq E_{k'}} d^{-2}\,\mathrm{e}^{\mathrm{i}t(E_n-E_{n'}+E_k-E_{k'})}.\end{aligned}$$

In a similar fashion

$$\begin{aligned}\langle \text{left}|\Phi_{(3412)}\rangle &= d^{-2}\,\mathrm{tr}_E(\mathrm{tr}_S^2|\psi\rangle\langle\psi|) \leq d^{-2}, \\ \langle\Phi_{(3412)}|\text{right}(t)\rangle &= \sum_{E_n\neq E_{n'}} d^{-2}\,\mathrm{e}^{\mathrm{i}t(2E_n-2E_{n'})}, \\ \langle \text{left}|\Phi_{(4321)}\rangle &= d^{-2}\sum_{s,e,s',e'}\langle se|se'\rangle\langle s'e'|s'e\rangle = d_E^{-1}, \\ \langle\Phi_{(4321)}|\text{right}(t)\rangle &= \sum_{E_n\neq E_{n'}} d^{-2} \leq 1, \\ \langle \text{left}|\Phi_{(1432)}\rangle &= d^{-2}\,\mathrm{tr}_E(\mathrm{tr}_S^2|\psi\rangle\langle\psi|) \leq d^{-2}, \\ \langle\Phi_{(1432)}|\text{right}(t)\rangle &= \sum_{E_n\neq E_{n'}\neq E_k} d^{-2}\,\mathrm{e}^{\mathrm{i}t(E_n+E_k-2E_{n'})}. \\ \langle \text{left}|\Phi_{(3214)}\rangle &= d^{-2}\,\mathrm{tr}_S(\mathrm{tr}_E^2|\psi\rangle\langle\psi|) \leq d^{-2}, \\ \langle\Phi_{(3214)}|\text{right}(t)\rangle &= \sum_{E_{n'}\neq E_n\neq E_{k'}} d^{-2}\,\mathrm{e}^{\mathrm{i}t(2E_n+E_{n'}-E_{k'})}. \\ \langle \text{left}|\Phi_{(1324)}\rangle &= (d_E d)^{-1} \\ \langle\Phi_{(1324)}|\text{right}(t)\rangle &= \sum_{E_{n'}\neq E_n\neq E_{k'}} d^{-2}\,\mathrm{e}^{\mathrm{i}t(E_n-E_{k'})},\end{aligned}$$

and for $\pi = (4231)$ we obtain the same results as for $\pi = (1324)$. Note that for $\pi = (3412), (4321), (1432), (3214), (1324), (4231)$ we have $|\langle \text{left}|\Phi_\pi\rangle\langle\Phi_\pi|\text{right}(t)\rangle| \leq d_E^{-1}$. The fact that $\||\text{left}\rangle\| = d$ and $\||\text{right}(t)\rangle\| \leq d^2$ implies $|\langle\text{left}|A|\text{right}(t)\rangle| \leq d^3$. Using these two facts, the chain of equalities (18), and inequality (19), we obtain

$$\left\langle \mathrm{tr}_S(\Omega_t[\psi]^2)\right\rangle_U \leq \langle \text{left}|\Phi_{(1234)}|\text{right}(t)\rangle + \frac{6}{d_E} + d^3\lambda^C$$

Let ρ be the not-necessarily-pure initial state ($\rho = \sum_i p_i\psi_i$ where each ψ_i is pure). Any real-valued random variable X satisfies $\langle X\rangle \leq \langle X^2\rangle^{1/2}$. Using the above, the triangular inequality, and (17), we obtain

$$\begin{aligned}&\left\langle \|\rho_S(t)-\bar\rho_S\|_1\right\rangle_U \\ &\leq \sum_i p_i \left\langle \|\Omega_t[\psi_i]\|_1\right\rangle_U \\ &\leq \sum_i p_i \left\langle d_S\,\mathrm{tr}_S(\Omega[\psi_i]^2)\right\rangle_U^{1/2} \\ &\leq d_S^{1/2}\Big(\langle \text{left}|\Phi_{(1234)}|\text{right}(t)\rangle + \frac{6}{d_E} + d^3\lambda^C\Big)^{1/2} \qquad (20)\end{aligned}$$

Define $w = (\sum_{E_n=E_{n'}} 1)$ and note that $w \leq gd$. Direct calculation shows

$$\begin{aligned}&\langle \text{left}|\Phi_{(1234)}|\text{right}(t)\rangle \\ &\leq \sum_{E_n\neq E_{n'}}\sum_{E_k\neq E_{k'}} d^{-4}\,\mathrm{e}^{\mathrm{i}t(E_n-E_{n'}+E_k-E_{k'})} \\ &= |\tilde\mu(t)|^4 + w^2d^{-4} - 2\,w\,d^{-2}\,|\tilde\mu(t)|^2 \\ &\leq |\tilde\mu(t)|^4 + g^2d^{-2}, \qquad (21)\end{aligned}$$

which can be fed in (20) giving

$$\begin{aligned}&\left\langle \|\rho_S(t)-\bar\rho_S\|_1\right\rangle_U \\ &\leq d_S^{1/2}\Big(\,|\tilde\mu(t)|^4 + \frac{g^2}{d^2} + \frac{6}{d_E} + d^3\,2^{-\frac{\alpha C}{N}}\Big)^{1/2}.\end{aligned}$$

We conclude the proof of Result 1 with a simple probabilistic argument. Let X be a random variable taking positive values such that $\langle X\rangle \leq x_0$. If $\epsilon = \mathrm{prob}\{X > x\}$ then $(1-\epsilon)0 + \epsilon\,x \leq \langle X\rangle$, therefore $x \leq x_0/\epsilon$.

Proof of Result 2

By applying the previous simple probabilistic argument to (20), we obtain that almost all hamiltonians with complexity C satisfy

$$\begin{aligned}&\|\rho_S(t)-\bar\rho_S\|_1 \qquad (22)\\ &\leq \frac{d_S^{1/2}}{\epsilon}\left(\langle \text{left}|\Phi_{(1234)}|\text{right}(t)\rangle + d^3\,2^{-\frac{\alpha C}{N}} + \frac{6}{d_E}\right)^{1/2}.\end{aligned}$$

Note that

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\langle \text{left}|\Phi_{(1234)}|\text{right}(t)\rangle \qquad (23)\\ &= \sum_{E_n\neq E_{n'}}\sum_{E_k\neq E_{k'}} d^{-4}\delta(E_n-E_{n'}+E_k-E_{k'}) \;\leq\; \frac{g}{d}\,,\end{aligned}$$

where $\delta(E_n - E_{n'} + E_k - E_{k'})$ is a Kronecker delta. This bound, inequality (22), and the convexity of the square

root, imply

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\|\rho_S(t)-\bar{\rho}_S\|_1 \\
&\leq \frac{d_S^{1/2}}{\epsilon}\Big(\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\langle \text{left}|\Phi_{(1234)}|\text{right}(t)\rangle \\
&+ d^3\,2^{-\frac{\alpha C}{N}}+\frac{4}{d_E}\Big)^{1/2} \\
&\leq \frac{1}{\epsilon}\left(d^4\,2^{-\frac{\alpha C}{N}}+\frac{g}{d_E}+\frac{6\,d_S}{d_E}\right)^{1/2}. \qquad (24)
\end{aligned}$$

This shows Result 2.

Proof of Result 3

Result 3 can be proven by following the same steps as in the proof of Result 1, but replacing the average over the Haar measure by the aveage over circuits with complexity C. Let $\mathcal{K}(C)$ be the set of N-qubit circuits with C gates, from a particular universal gate set. The average of a function $F(U)$ over $U\in\mathcal{K}(C)$ is denoted by $\langle F(U)\rangle_{U\in\mathcal{K}(C)}$. It is shown in [25] that

$$\begin{aligned}
&\Big\langle U\otimes U\otimes U\otimes U\otimes \bar{U}\otimes\bar{U}\otimes\bar{U}\otimes\bar{U}\Big\rangle_{U\in\mathcal{K}(C)} \\
&\leq \sum_\pi |\Phi_\pi\rangle\langle\Phi_\pi| + \lambda^C A, \qquad (25)
\end{aligned}$$

where $\lambda = 1-\alpha\ln 2/N$ and the constant $\alpha>0$ depends on the universal gate set. The hermitian matrix A satisfies $\|A\|_\infty\leq 1$ and $A|\Phi_\pi\rangle = 0$ for all π. The range of π and the vectors $|\Phi_\pi\rangle$ are defined in the proof of Result 1.

Calculation of $|\tilde{\mu}(t)|$

Spectrum of a random matrix

In this section we calculate the convergence function $|\tilde{\mu}(t)|$ for the spectrum of a random hamiltonian H, generated by the probability distribution $P(H)$ corresponding to the gaussian unitary ensemble [24]. According to this, each matrix element H_{ij} is an independent random variable; the elements in the diagonal $H_{ii}\in\mathbb{R}$ have probability density $P(H_{ii}) = \pi^{-1/2}\,\mathrm{e}^{-H_{ii}^2}$; the elements not in the diagonal $H_{ij}=\overline{H}_{ji}\in\mathbb{C}$ have probability density $P(H_{ij}) = (2/\pi)\,\mathrm{e}^{-2|H_{ij}|^2}$. This is equivalent to say that the diagonalizing-unitary U of H follows the uniform distribution over unitaries (the Haar measure [22]), and independently, the spectrum of H follows the probability density

$$P(E_1,\ldots,E_d) = \alpha\,\mathrm{e}^{-\sum_{i=1}^d E_i^2}\prod_{1\leq i<j\leq d}|E_i-E_j|^2 \qquad (26)$$

where $E_i\in(-\infty,\infty)$, and α is a normalization constant. That is, eigenvalues and eigenvectors are independent random variables.

Since, what appears in expression (4) is $|\tilde{\mu}(t)|$ to the fourth power, we are going to calculate the average

$$\big\langle|\tilde{\mu}(t)|^4\big\rangle_H = \int dE_1\cdots dE_d\,P(E_1,\ldots,E_d)\,|\tilde{\mu}(t)|^4.$$

A standard trick within random matrix theory is that, with probability almost one, the value of $|\tilde{\mu}(t)|^4$ for a randomly chosen spectrum is very close to the above average.

For the next, it is useful to define the n-point correlation-function

$$R_n(E_1,...,E_n) = \frac{d!}{(d-n)!}\int dE_{n+1}\cdots dE_d\,P(E_1,\ldots,E_d)$$

where $0<n<d$ [24]. The sum

$$d^4\,\big\langle|\tilde{\mu}(t)|^4\big\rangle_H = \sum_{ijkl}\Big\langle \mathrm{e}^{\imath t(E_i-E_j+E_k-E_l)}\Big\rangle_H$$

can be split in the four terms where i,j,k,l are: (i) all different, (ii) two of them equal; (iii) three of them equal or two pairs equal, and (iv) all equal. These four terms are separated in

$$\begin{aligned}
d^4\,\big\langle|\tilde{\mu}(t)|^4\big\rangle_H &= \int dE_1\,dE_2\,dE_3\,dE_4\;R_4(E_1,E_2,E_3,E_4)\;\mathrm{e}^{\mathrm{i}t(E_1-E_2+E_3-E_4)} \\
&+ \int dE_1\,dE_2\,dE_3\;R_3(E_1,E_2,E_3)\left[\mathrm{e}^{\mathrm{i}t(2E_1-E_2-E_3)}+\mathrm{e}^{-\mathrm{i}t(2E_1-E_2-E_3)}+4\,\mathrm{e}^{\mathrm{i}t(E_1-E_2)}\right] \\
&+ \int dE_1\,dE_2\;R_2(E_1,E_2)\left[\mathrm{e}^{\mathrm{i}t2(E_1-E_2)}+4\,\mathrm{e}^{\mathrm{i}t(E_1-E_2)}\right] \;+\; \left[2d^2-d\right]. \qquad (27)
\end{aligned}$$

It is shown in [24] that the n-point correlation functions can be written as

$$R_n(E_1,\ldots,E_n) = \det[K(E_i,E_j)]_{i,j=1,\ldots,n} \qquad (28)$$

where

$$K(E_i, E_j) = \sum_{k=0}^{d-1} \varphi_k(E_i)\, \varphi_k(E_j) \ , \tag{29}$$

where $\varphi_k(x)$ are the eigenfunctions of the quantum harmonic oscillator. To see how the determinant works in (28) consider the example

$$R_2(E_1, E_2) = K(E_1, E_1)\, K(E_2, E_2) - K(E_1, E_2)^2 \ .$$

When substituting (28) in (27), we obtain products of objects of the form

$$\int dE\, K(E, E)\, \mathrm{e}^{\mathrm{i}tE} \tag{30}$$

and

$$\begin{aligned} &\int dE_1 \cdots dE_n\, K(E_1, E_2) \cdots K(E_n, E_1)\, \mathrm{e}^{\mathrm{i}(t_1 E_1 + \cdots + t_n E_n)} \\ &\quad = \mathrm{tr}\big[P\, \mathrm{e}^{\mathrm{i}t_1 X}\, P\, \mathrm{e}^{\mathrm{i}t_2 X}\, \cdots P\, \mathrm{e}^{\mathrm{i}t_n X}\big] \end{aligned} \tag{31}$$

for $n = 2, 3, 4$—where $P = \sum_{k=0}^{d-1} |\varphi_k\rangle\langle\varphi_k|$ is the projector onto the d-dimensional lower-energy subspace of the harmonic oscillator, and X is the position operator. For any pair of bounded operators A, B, the Cauchy-Schwartz inequality tells

$$\big|\mathrm{tr}[A^\dagger B]\big| \leq \sqrt{\mathrm{tr}[A^\dagger A]\, \mathrm{tr}[B^\dagger B]} \ . \tag{32}$$

This can be used to bound (31) for the case $n = 4$:

$$\begin{aligned} &\Big|\mathrm{tr}\big[\big(P\, \mathrm{e}^{\mathrm{i}t_1 X} P\, \mathrm{e}^{\mathrm{i}t_2 X}\big)\big(P\, \mathrm{e}^{\mathrm{i}t_3 X} P\, \mathrm{e}^{\mathrm{i}t_4 X}\big)\big]\Big| \\ &\leq \sqrt{\mathrm{tr}\big[\mathrm{e}^{-\mathrm{i}t_2 \hat{X}} \hat{P}\, \mathrm{e}^{-\mathrm{i}t_1 \hat{X}} \hat{P}^2\, \mathrm{e}^{\mathrm{i}t_1 \hat{X}} \hat{P}\, \mathrm{e}^{\mathrm{i}t_2 \hat{X}}\big] \mathrm{tr}\big[\hat{P}\, \mathrm{e}^{-\mathrm{i}t_3 \hat{X}} \hat{P}\, \mathrm{e}^{\mathrm{i}t_3 \hat{X}}\big]} \\ &\leq \sqrt{\sqrt{\mathrm{tr}\big[\hat{P}\big]\mathrm{tr}\big[\hat{P}\big]}\sqrt{\mathrm{tr}\big[\hat{P}\big]\mathrm{tr}\big[\hat{P}\big]}} \leq d. \end{aligned} \tag{33}$$

By setting $t_4 = 0$ and $t_3 = 0$, we obtain the same bound for $n = 3, 2$. It is shown in [24] that in the large-d limit we have

$$K(E, E) \approx \begin{cases} \frac{1}{\pi}\sqrt{2d - E^2} & \text{if} \quad |E| \leq \sqrt{2d} \\ 0 & \text{if} \quad |E| > \sqrt{2d} \end{cases} \ . \tag{34}$$

This is called "the semi-circle law". In this limit, the integral (30) can be evaluated

$$\int dE\, K(E, E)\, \mathrm{e}^{\mathrm{i}tE} \ \approx \ d\, \frac{2\, J_1(t\sqrt{2d})}{t\sqrt{2d}} \ \leq \ d \ , \tag{35}$$

where J_1 is the Bessel function of the first kind. When substituting (28) in (27), we obtain several terms, each being a product of objects of the form (30) or (31). According to (33) and (35), each of these factors (30) or (31) is bounded by d. Since $\big\langle |\tilde{\mu}(t)|^4 \big\rangle_H$ is equal to (27) divided by d^4, all terms are of order $1/d$ or smaller, except for the single term with a four-fold product of (30). This implies that in the large-d limit we have

$$\big\langle |\tilde{\mu}(t)|^4 \big\rangle_H \ \approx \ \left(\frac{2\, J_1(t\sqrt{2d})}{t\sqrt{2d}}\right)^4 . \tag{36}$$

Spectrum of the Ising model

The eigenenergies are parametrized by the vectors $n = (n_1, \ldots, n_N)$, where $n_k \in \{0, 1\}$ are the occupation numbers of the energy eigenmodes:

$$E_n \ = \ \sum_{k=1}^{N} n_k\, \omega(2\pi k/N), \tag{37}$$

$$\omega(\phi) \ = \ \sqrt{(h - \cos\phi)^2 + \sin^2\phi} \ . \tag{38}$$

Then

$$\begin{aligned} |\tilde{\mu}(t)|^2 \ &= \ 2^{-2N} \prod_{k=1}^{N} \left| \sum_{n_k=0}^{1} \mathrm{e}^{\mathrm{i}t n_k \omega(2\pi k/N)} \right|^2 \\ &= \ 2^{-N} \prod_{k=1}^{N} \left(1 + \cos[t\omega(2\pi k/N)]\right) \\ &\approx \ 2^{-N} \exp\left[\frac{N}{2\pi} \int_0^{2\pi} d\phi\, \ln(1 + \cos[t\, \omega(\phi)])\right], \end{aligned} \tag{39}$$

where the approximation holds for large N. Let us analyze this expression in the small-t and large-t limits. For $t \ll 1$ we have, up to fourth-order terms,

$$\ln(1 + \cos[t\, \omega(\phi)]) \approx \ln 2 - \big(1 + h^2 - 2h\cos[\phi]\big)\, \frac{t^2}{4},$$

which gives

$$|\tilde{\mu}(t)|^2 \approx \mathrm{e}^{-t^2 N(1+h^2)/4}. \tag{40}$$

For $t \gg 1$ we have

$$\begin{aligned} |\tilde{\mu}(t)|^2 \ &\leq \ 2^{-N} \exp\left[N \ln\left(1 + \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \cos[t\, \omega(\phi)]\right)\right] \\ &\approx \ 2^{-N} \exp[N \ln(1)] = 2^{-N}, \end{aligned} \tag{41}$$

where the inequality follows from the convexity of the logarithm, and the approximation holds when the integrand oscillates heavily, that is when $t \gg 1$. In the large-N limit, the largest eigenvalue is

$$E_{\max} = \frac{N}{2\pi} \int_0^{2\pi} d\phi\, \sqrt{(h - \cos\phi)^2 + \sin^2\phi} \ , \tag{42}$$

which is proportional to N.